ON THE RELATIVE PERIODIC MOTIONS OF A PENDULUM

(OB OTNOSITEL'NYKH PERIODICHESKIKH DVIZHENIIAKH MAIATNIKA)

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Considered are the properties of the relative periodic motions of a rigid body suspended on an elastic string in a uniformly rotating vertical plane. The relative periodic motions of a mathematical pendulum with an elastic string were considered in [1] and [2].

1. Let us denote by Oxy a system of coordinates rotating uniformly with respect to the downward Oy-axis, relative to which the motion of the rigid body will be studied. The elastic string, considered as a linear massless spring with an elastic constant c, is attached at a point O_1 (See Fig.1) where $\overline{OO_1} = h$. The angle of deflection of the string from the vertical axis $O_1 y_1$ will be denoted by φ_1 and its length by ρ. A body with a mass m is suspended from the spring at point O_2. The angle between the straight line passing through O_2 and the center of gravity C of the body and the vertical will be denoted as φ_2, and let us assume $O_2C = a$. Choosing a body system of coordinates $C\xi\eta\zeta$, such that $C\eta$ passes through O_2, $C\xi$ is orthogonal to $C\eta$ and lying in the plane Oxy, while $C\zeta$ is orthogonal to Oxy.

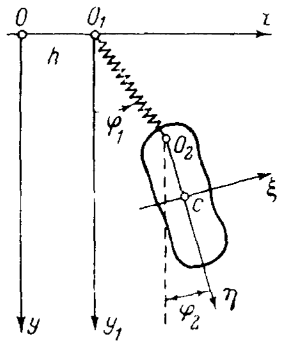

Fig. 1

We assume that the axes of the system $C\xi\eta\zeta$ are the principal axes of inertia of the body. The principal moments of inertia with respect to the axes $C\xi$, $C\eta$ and $C\zeta$ will be denoted by $J_\nu\ (\nu = 1, 2, 3)$, respectively.

The kinetic energy of the system in its absolute motion, according to the Koenig theorem, is

$$T = {}^1/_2\, m v_c^2 + {}^1/_2\, [J_3 \dot\varphi_2^2 + \omega^2 (J_1 \sin^2\varphi_2 + J_2 \cos^2\varphi_2)]$$

where v_c is the absolute velocity of point C. In view of the fact that the coordinates of the center of gravity of the body relative to the Oxy system are defined by Expressions

$$x_c = h + \rho \sin\varphi_1 + a \sin\varphi_2,$$

$$y_c = \rho\cos\varphi_1 = a\cos\varphi_2$$

and that

$$v_c^2 = \dot x_c^2 + \dot y_c^2 + \omega^2 x_c^2$$

the expression for the kinetic energy becomes

$$T = {}^1/_2\, m\, [\rho^{\cdot 2} + \rho^2\varphi_1^{\cdot 2} + a\varphi_2^{\cdot 2} + 2a\rho^\cdot\varphi_2^\cdot \sin(\varphi_1 - \varphi_2) + 2a\rho\varphi_1^\cdot\varphi_2^\cdot \cos(\varphi_1 - \varphi_2) + \omega^2 (h + \rho \sin\varphi_1 + a \sin\varphi_2)^2] + {}^1/_2\, [J_3\varphi_2^{\cdot 2} + \omega^2 (J_1 \sin^2\varphi_2 + J_2 \cos^2\varphi_2)] \qquad (1.1)$$

The potential energy of the system i.e. the sum of the energy of elastic deformation of the string and the gravitational energy, is given by the Expression

$$\Pi = {}^1/_2\, c\, (\rho - l)^2 - mg\, (\rho \cos\varphi_1 + a \cos\varphi_2) \qquad (1.2)$$

where l is the free length of the string.

Bearing in mind (1.1) and (1.2) we obtain, according to the Lagrange equations, the system of differential equations of motion

$$\rho^{\cdot\cdot} + \rho\,(k^2 - \varphi_1^{\cdot 2} - \omega^2 \sin^2\varphi_1) + a\varphi_2^{\cdot\cdot} \sin(\varphi_1 - \varphi_2) - a\varphi_2^{\cdot 2}\cos(\varphi_1 - \varphi_2) - \omega^2\ (h + a\sin\varphi_2)\sin\varphi_1 - g\cos\varphi_1 - k^2 l = 0$$

$$\rho\varphi_1^{\cdot\cdot} + 2\rho^\cdot\varphi_1^\cdot + a\varphi_2^{\cdot\cdot}\cos(\varphi_1 - \varphi_2) + a\varphi_2^\cdot \sin(\varphi_1 - \varphi_2) - \omega^2 (h + \rho\sin\varphi_1 + a\sin\varphi_2)\cos\varphi_1 + g\sin\varphi_1 = 0 \qquad (1.3)$$

$$l_1\varphi_2^{\cdot\cdot} + (\rho\varphi_1^{\cdot\cdot} + 2\rho^\cdot\varphi_1^\cdot)\cos(\varphi_1 - \varphi_2) - (\rho^{\cdot\cdot} - \rho\varphi_1^{\cdot 2})\sin(\varphi_1 - \varphi_2) - \omega^2 (h + \rho\sin\varphi_1 + a\sin\varphi_2)\cos\varphi_2 + [(J_2 - J_1)/2am]\sin 2\varphi_2 + g\sin\varphi_2 = 0$$

where

$$l_1 = \frac{1}{a}\left(a^2 + \frac{J_3}{m}\right), \qquad k^2 = \frac{c}{m}$$

Here l_1 is the derived length of the physical pendulum (body) relative to the point O_2 .

Let us consider the oscillations of the system near the position of relative equilibrium. Let us assume

$$J_1 = J_2 \qquad (1.4)$$

It can be proved that if (1.4) is valid, then for the state of relative equilibrium the angles φ_{10} and φ_{20} are equal to each other and, consequently, the following substitution can be made:

$$\rho(t) = b + \xi(t), \qquad \varphi_1(t) = \varphi_0 + \varphi(t), \qquad \varphi_2(t) = \varphi_0 + \psi(t) \qquad (1.5)$$

Here φ_0 is the value of the angles φ_{10} and φ_{20}, and b is the length of the pendulum string in relative equilibrium. The quantities φ_0 and b are determined by the equalities

$$k^2 (b - l) = \omega^2 (h + b\sin\varphi_0 + a\sin\varphi_0)\sin\varphi_0 + g\cos\varphi_0 \qquad (1.6)$$

$$g\sin\varphi_0 = \omega^2 (h + b\sin\varphi_0 + a\sin\varphi_0)\cos\varphi_0$$

Bearing in mind (1.4) and (1.6), by the substitution of (1.5) into (1.3), we obtain

$$\xi^{\cdot\cdot} + (k^2 - \omega^2\sin\varphi_0)\,\xi - {}^1/_2\omega^2 b\sin 2\varphi_0\varphi - {}^1/_2\omega^2 a\sin 2\varphi_0\psi = f_1 + \ldots \qquad (1.7)$$

$$b\varphi^{\cdot\cdot} + a\psi^{\cdot\cdot} - \omega^2 a\cos^2\varphi_0\xi + [k^2 (b - l) - \omega^2 b\cos^2\varphi_0]\,\varphi - \omega^2 a\cos^2\varphi_0\psi = f_2 + \ldots$$

$$l_1\psi^{\cdot\cdot} + b\varphi^{\cdot\cdot} - {}^1/_2\omega^2\sin 2\varphi_0\xi - \omega^2 b\cos^2\varphi_0\varphi + [k^2 (b - l) - \omega^2 a\cos^2\varphi_0]\,\psi = f_3 + \ldots$$

where

$$f_1 = b\varphi^{\cdot 2} - a(\varphi - \psi)\psi^{\cdot\cdot} + \omega^2\sin 2\varphi_0\xi\varphi + \omega^2 a\cos^2\varphi_0\varphi\psi - {}^1/_2\omega^2 a\sin^2\varphi_0\psi^2 + a\psi^{\cdot 2} + {}^1/_2\,(2\omega^2 b\cos 2\varphi_0 - a\omega^2\sin^2\varphi_0 - \omega^2 h\sin\varphi_0 - g\cos\varphi_0)\,\varphi^2$$

$$f_2 = -\xi\varphi^{\cdot\cdot} - 2\xi^\cdot\varphi^\cdot + \omega^2\cos 2\varphi_0\xi\varphi - {}^1/_2\omega^2 a\sin 2\varphi_0\varphi\psi - {}^3/_4\omega^2 b\sin 2\varphi_0\varphi^2 - {}^1/_2\omega^2 a\sin 2\varphi_0\psi^2$$

$$f_3 = -\xi\varphi^{\cdot\cdot} - (\varphi - \psi)\,\xi^{\cdot\cdot} - 2\xi^\cdot\varphi^\cdot + \omega^2\cos^2\varphi_0\xi\varphi - {}^1/_2\omega^2 b\sin 2\varphi_0\varphi\psi - \omega^2\sin^2\varphi_0\xi\psi - {}^1/_4\omega^2 b\sin 2\varphi_0\varphi^2 - {}^3/_4\omega^2\sin 2\varphi_0\psi^2 \qquad (1.8)$$

Neglecting the nonlinear terms we obtain the system

$$
\begin{aligned}
\xi^{\cdot\cdot} + a_{11}\xi + a_{12}\varphi + a_{13}\psi &= 0 \\
b\varphi^{\cdot\cdot} + a\psi^{\cdot\cdot} + b_{11}\xi + b_{12}\varphi + b_{13}\psi &= 0 \\
l_1\psi^{\cdot\cdot} + b\varphi^{\cdot\cdot} + c_{11}\xi + c_{12}\varphi + c_{13}\psi &= 0
\end{aligned} \tag{1.9}
$$

where

$$
\begin{array}{lll}
a_{11} = k^2 - \omega^2 \sin^2\varphi_0, & b_{11} = -\omega^2 a \cos^2\varphi_0, & c_{11} = -{}^1/_2\omega^2 \sin 2\varphi_0 \\
a_{12} = -{}^1/_2\omega^2 b \sin 2\varphi_0, & b_{12} = k^2(b-l) - \omega^2 b \cos^2\varphi_0, & c_{12} = -\omega^2 b \cos^2\varphi_0 \\
a_{13} = -{}^1/_2\omega^2 a \sin 2\varphi_0, & b_{13} = -\omega^2 a \cos^2\varphi_0, & c_{13} = k^2(b-l) - \omega^2 a \cos^2\varphi_0
\end{array} \tag{1.10}
$$

The fundamental equation of the system will be

$$
\begin{vmatrix}
a_{11} + r^2 & a_{12} & a_{13} \\
b_{11} & b_{12} + br^2 & b_{13} + ar^2 \\
c_{11} & c_{12} + br^2 & c_{13} + l_1 r^2
\end{vmatrix} = 0 \tag{1.11}
$$

The condition that Equation (1.11) would have purely imaginary roots is reduced to the inequality

$$
\begin{aligned}
&k^2 \left[g.\cos\varphi_0 + \omega^2 \left(h \sin\varphi_0 + l \sin^2\varphi_0 - b\cos^2\varphi_0 - a\cos 2\varphi_0\right)\right] + \\
&\quad + \omega^4 b \cos^2\varphi_0 \sin\varphi_0 (\sin\varphi_0 - a\cos\varphi_0) > 0
\end{aligned} \tag{1.12}
$$

Under condition (1.12) the system has three pairs of purely imaginary roots. On the basis of the general theory of linear equations with constant coefficients, the system (1.7) can be transformed into a form similar to the system [3] (p.435)

$$
\frac{dx}{dt} = -ry + X, \qquad \frac{dy}{dt} = rx + Y
$$

$$
\frac{dx_s}{dt} = b_{s1}x_1 + \ldots + b_{sm}x_m + a_s x + b_s y + X_s \qquad (s = 1, \ldots, m;\ m = 4)
$$

where $\pm ir$ is any pair of purely imaginary roots.

The proposed system may be considered as a Liapunov system relative to such a pair of purely imaginary roots. On the basis of the theorem [3] (p.442), it may be asserted that the system permits a periodic solution dependent on an arbitrary parameter. This parameter is the initial value ξ of the quantity ξ.

The "basic" pair of purely imaginary roots can be any pair and therefore this system permits three periodic solutions.

2. Now let us study the properties of periodic motions near the position of relative equilibrium which have an approximate period $2\pi/r_1$.

Let the period of the solution be of the form

$$
T_1 = \frac{2\pi}{r_1}(1 + \delta_1\lambda + \delta_2\lambda^2 + \ldots) \tag{2.1}
$$

where $\delta_1, \delta_2, \ldots$ are constants subject to determination and $\lambda = \alpha\xi$.

We introduce the variable τ in place of the variable t into the equations by means of the substitution

$$
t = \tau(1 + \delta_1\lambda + \delta_2\lambda^2 + \ldots) \tag{2.2}
$$

Then the problem is reduced to finding the periodic solutions with the period T_1 of the system

$$\begin{aligned} \xi^{\cdot\cdot} + a_{11}\xi + a_{12}\varphi + a_{13}\psi &= f_1^* + \ldots \\ b\varphi^{\cdot\cdot} + a\psi^{\cdot\cdot} + b_{11}\xi + b_{12}\varphi + b_{13}\psi &= f_2^* + \ldots \\ l_1\psi^{\cdot\cdot} + b\varphi^{\cdot\cdot} + c_{11}\xi + c_{12}\varphi + c_{13}\psi &= f_3^* + \ldots \end{aligned} \tag{2.3}$$

where f_1^*, f_2^* and f_3^* are obtained from (1.8). The derivatives here and in the following are with respect to τ.

The solutions of the system (2.3) are analytic with respect to λ, and therefore they will be sought in the form of the series

$$\begin{aligned} \xi(\tau) &= \lambda\xi_1(\tau) + \lambda^2\xi_2(\tau) + \lambda^3\xi_3(\tau) + \ldots \\ \varphi(\tau) &= \lambda\varphi_1(\tau) + \lambda^2\varphi_2(\tau) + \lambda^3\varphi_3(\tau) + \ldots \\ \psi(\tau) &= \lambda\psi_1(\tau) + \lambda^2\psi_2(\tau) + \lambda^3\psi_3(\tau) + \ldots \end{aligned} \tag{2.4}$$

where $\xi_\nu, \varphi_\nu, \psi_\nu$ $(\nu = 1, 2, 3)$ are periodic functions of τ of the period $2\pi/r_1$ (here φ_1 and φ_2 are periodic functions of τ and have not the same meaning as in Section 1) satisfying the initial conditions

$$\xi_1(0) = 1, \quad \varphi_1(0) = k_1, \quad \psi_1(0) = k_2, \quad \xi_\nu^{\cdot}(0) = \varphi_\nu^{\cdot}(0) = \psi_\nu^{\cdot}(0) = 0 \qquad (\nu = 1, 2, 3) \tag{2.5}$$

Substituting (2.4) into (2.3) and equating the coefficients of like powers of λ we obtain a system of equations for determination of ξ_ν, φ_ν and ψ_ν $(\nu = 1,2,3)$.

For the functions ξ_1, φ_1 and ψ_1 there results a basic system (1.9) which has the obvious solution

$$\xi_1(\tau) = \cos r_1\tau, \qquad \varphi_1(\tau) = k_1 \cos r_1\tau, \qquad \psi_1(\tau) = k_2 \cos r_1\tau \tag{2.6}$$

where

$$k_1 = \frac{(a_{11} - r_1^2)(b_{13} - ar_1^2) - a_{13}c_{11}}{a_{13}(b_{12} - br_1^2) - a_{12}(b_{13} - ar_1^2)}, \qquad k_2 = \frac{(a_{11} - r_1^2)(b_{12} - br_1^2) - a_{12}c_{11}}{a_{13}(b_{12} - br_1^2) - a_{12}(b_{13} - ar_1^2)}$$

Furthermore, we have

$$\begin{aligned} \xi_2^{\cdot\cdot} + a_{11}\xi_2 + a_{12}\varphi_2 + a_{13}\psi_2 &= F_0 - 2\delta_1F_1\cos r_1\tau + F_2\cos 2r_1\tau \\ b\varphi_2^{\cdot\cdot} + a\psi_2^{\cdot\cdot} + b_{11}\xi_2 + b_{12}\varphi_2 + b_{13}\psi_2 &= G_0 - 2\delta_1G_1\cos r_1\tau + G_2\cos 2r_1\tau \\ l_1\psi_2^{\cdot\cdot} + b\varphi_2^{\cdot\cdot} + c_{11}\xi_2 + c_{12}\varphi_2 + c_{13}\psi_2 &= H_0 - 2\delta_1H_1\cos r_1\tau + H_2\cos 2r_1\tau \end{aligned} \tag{2.7}$$

where the expressions for F_ν, G_ν and H_ν are not derived because of their complexity.

The periodic solution (with the period $2\pi/r_1$) of the homogeneous system corresponding to (2.7) will be

$$\begin{aligned} \xi_{21} &= C_1\cos r_1\tau + D_1\sin r_1\tau, \qquad \varphi_{21} = k_1(C_1\cos r_1\tau + D_1\sin r_1\tau) \\ \psi_{21} &= k_2(C_1\cos r_1\tau + D_1\sin r_1\tau) \end{aligned} \tag{2.8}$$

where the unknown constants C_1 and D_1 will be determined later.

The particular solution of the system (2.7) is sought in the form

$$\begin{aligned} \xi_{22} &= P_0 + P_1\cos r_1\tau + P_2\cos 2r_1\tau, \qquad \varphi_{22} = Q_0 + Q_1\cos r_1\tau + Q_2\cos 2r_1\tau \\ \psi_{22} &= R_0 + R_1\cos r_1\tau + R_2\cos 2r_1\tau \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.7) we obtain for P_ν, Q_ν and R_ν $(\nu = 0,1,2)$ the systems

$$\begin{aligned} a_{11}P_0 + a_{12}Q_0 + a_{13}R_0 &= F_0 \\ b_{11}P_0 + b_{12}Q_0 + b_{13}R_0 &= G_0 \\ c_{11}P_0 + c_{12}Q_0 + c_{13}R_0 &= H_0 \end{aligned} \tag{2.10}$$

$$(a_{11} - r_1^2) P_1 + a_{12}Q_1 + a_{13}R_1 = -2\delta_1 F_1$$
$$b_{11}P_1 + (b_{12} - br_1^2) Q_1 + (b_{13} - ar_1^2) R_1 = -2\delta_1 G_1 \quad (2.11)$$
$$c_{11}P_1 + (c_{12} - br_1^2) Q_1 + (c_{13} - l_1 r_1^2) R_1 = -2\delta_1 H_1$$

$$(a_{11} - 4r_1^2) P_1 + a_{12}Q_2 + a_{13}R_2 = F_2$$
$$b_{11}P_2 + (b_{12} - 4br_1^2) Q_2 + (b_{13} - 4ar_1^2) R_2 = G_2 \quad (2.12)$$
$$c_{11}P_2 + (c_{12} - 4br_1^2) Q_2 + (c_{13} - 4l_1 r_1^2) R_2 = H_2$$

Thus, the problem is reduced to the determination of conditions for which these systems can be solved with respect to P_ν, Q_ν and R_ν.

Since 0 and $\pm 2ir_1$ are not the roots of Equations (1.11), the system (2.10) and (2.12) have a unique solution for P_0, Q_0, R_0 and P_2, Q_2, R_2 respectively.

The characteristic equation of the determinant for the system (2.11) coincides exactly with Equation (1.11) for which $r = \pm ir_1$ is a root. It is obvious now that for $\delta_1 = 0$ the system (2.11) has a solution $Q_1 = k_1 P_1$ and $R_1 = k_2 P_1$. Thus,

$$\xi_2 = P_0 + P_1' \cos r_1\tau + D_1 \sin r_1\tau + P_2 \cos 2r_1\tau$$
$$\varphi_2 = Q_0 + Q_1' \cos r_1\tau + k_1 D_1 \sin r_1\tau + Q_2 \cos 2r_1\tau \quad (2.13)$$
$$\psi_2 = R_0 + R_1' \cos r_1\tau + k_2 D_1 \sin r_1\tau + R_2 \cos 2r_1\tau$$

where

$$P_1' = P_1 + C_1, \qquad Q_1' = Q_1 + k_1 C_1, \qquad R_1' = R_1 + k_2 C_1$$

From condition (2.5) it follows directly that $D_1 = 0$, therefore we can write

$$\xi_2 = P_0 + P_1' \cos r_1\tau + P_2 \cos 2r_1\tau, \qquad \varphi_2 = Q_0 + Q_1' \cos r_1\tau + Q_2 \cos 2r_1\tau$$
$$\psi_2 = R_0 + R_1' \cos r_1\tau + R_2 \cos 2r_1\tau \quad (2.14)$$

The constants P_1', Q_1' and R_1' as well as δ_2 are determined from the periodicity condition of the functions ξ_3, φ_3 and ψ_3 which are not derived because of their bulkiness.

It is interesting to note that the coefficients of the λ terms in (2.4) represent particular solutions of the corresponding systems.

The periodic solutions of the system (1.7) are expressed approximately as

$$\xi = \lambda \cos r_1\tau + \lambda^2 (P_0 + P_1' \cos r_1\tau + P_2 \cos 2r_1\tau) + \ldots$$
$$\varphi = \lambda k_1 \cos r_1\tau + \lambda^2 (Q_0 + Q_1' \cos r_1\lambda + Q_2 \cos 2r_1\tau) + \ldots \quad (2.15)$$
$$\psi = \lambda k_2 \cos r_1\tau + \lambda^2 (R_0 + R_1' \cos r_1\tau + R_2 \cos 2r_1\tau) + \ldots$$

and the period of the motion is

$$T_1 = \frac{2\pi}{r_1} (1 + \delta_2\lambda^2 + \ldots)$$

The tension force K in the string of the pendulum is of the form

$$K = c(\rho - l)$$

or, taking into account (2.15)

$$K = c\,[b - l + \lambda \cos r_1\tau + \lambda^2 (P_0 + P_1' \cos r_1\tau + P_2 \cos 2r_1\tau) + \ldots]$$

Similarly, it can be established that periodic solutions also correspond to the roots $\pm ir_2$ and $\pm ir_3$

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